

Two statements on path systems related to quantum minors

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Abstract

In [4] we gave a complete combinatorial characterization of homogeneous quadratic identities for minors of quantum matrices. It was obtained as a consequence of results on minors of matrices of a special sort, the so-called *path matrices* Path_G generated by paths in special planar directed graphs G .

In this paper we prove two assertions that were stated but left unproved in [4]. The first one says that any minor of Path_G is determined by a system of disjoint paths, called a *flow*, in G (generalizing a similar result of Lindström's type for the path matrices of Cauchon graphs in [2]). The second, more sophisticated, assertion concerns certain transformations of pairs of flows in G .

Keywords: quantum matrix, planar graph, Cauchon diagram, Lindström Lemma

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1 Introduction

This paper is a supplement to [4] where we developed a graph theoretic construction (borrowing an idea of [2]) that was used as the main tool to obtain a complete combinatorial characterization for the variety of homogeneous quadratic identities on minors of quantum matrices.

(Recall that when speaking of the *algebra of $m \times n$ quantum matrices*, one means the quantized coordinate ring $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ of $m \times n$ matrices over a field \mathbb{K} , where q is a nonzero element of \mathbb{K} . In other words, one considers the \mathbb{K} -algebra generated by indeterminates x_{ij} ($i \in [m]$, $j \in [n]$) satisfying Manin's relations [7]: for $i < \ell \leq m$ and $j < k \leq n$,

$$\begin{aligned} x_{ij}x_{ik} &= qx_{ik}x_{ij}, & x_{ij}x_{\ell j} &= qx_{\ell j}x_{ij}, \\ x_{ik}x_{\ell j} &= x_{\ell j}x_{ik} & \text{and} & \quad x_{ij}x_{\ell k} - x_{\ell k}x_{ij} = (q - q^{-1})x_{ik}x_{\ell j}. \end{aligned} \tag{1.1}$$

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Hereinafter for a positive integer n' , $[n']$ denotes $\{1, 2, \dots, n'\}$. Another useful algebraic construction is the $m \times n$ *quantum affine space*, which is the \mathbb{K} -algebra generated by indeterminates t_{ij} ($i \in [m], j \in [n]$) subject to “simpler” commutation relations:

$$\begin{aligned} t_{ij}t_{i'j'} &= qt_{i'j'}t_{ij} && \text{if either } i = i' \text{ and } j < j', \text{ or } i < i' \text{ and } j = j', \\ &= t_{i'j'}t_{ij} && \text{otherwise.} \end{aligned} \quad (1.2)$$

In this paper we prove two auxiliary theorems that were essentially used, but left unproved, in [4] (namely, Theorems 3.1 and 4.4 there). They concern the class of edge-weighted planar graphs introduced in [4] (under the name of “grid-shaped graphs”); in this paper we call them *SE-graphs*. A special case of these graphs is formed by the *Cauchon graphs* introduced in [2] in connection with the *Cauchon diagrams* of [1]. The first theorem, viewed as a quantum analog of Lindström Lemma, is a direct extension to the SE-graphs G of the corresponding result established for Cauchon graphs in [2]. It considers a matrix in which each entry is represented as the sum of weights of paths connecting a certain pair of vertices of G , called the *path matrix* of G and denoted by Path_G . The theorem asserts that any (quantized) minor of Path_G can be expressed via systems of disjoint paths of G connecting corresponding sets of vertices. We refer to a system of this sort as a *flow* in G .

The proof of the main result in [4] (which can be regarded as a quantum analog of a characterization of quadratic identities for the commutative case in [5]) is based on a method of handling certain pairs of flows, called *double flows*, in an SE-graph G . An important ingredient of that proof is a transformation of a double flow (ϕ, ϕ') into another double flow (ψ, ψ') by use of an *ordinary exchange operation*. The second theorem that we are going to prove in this paper says that under such a transformation the weight of a current double flow is multiplied by q or q^{-1} .

The paper is organized as follows. Section 2 contains basic definitions and formulates the first theorem. Section 3 describes exchange operations on double flows and formulates the second theorem. Section 4 elaborates technical tools needed to prove the theorems. It considers certain paths P, Q in G and describes possible relations between the weights of the ordered pairs (P, Q) and (Q, P) ; this is close to a machinery in [2, 3]. The announced first and second theorems are proved in Sections 5 and 6, respectively.

2 Preliminaries

We start with basic definitions and some elementary properties.

Paths in graphs. Throughout, by a *graph* we mean a directed graph. A *path* in a graph $G = (V, E)$ (with vertex set V and edge set E) is a sequence $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ such that each e_i is an edge connecting vertices v_{i-1}, v_i . An edge e_i is called *forward* if it is directed from v_{i-1} to v_i , denoted as $e_i = (v_{i-1}, v_i)$, and *backward* otherwise (when $e_i = (v_i, v_{i-1})$). The path P is called *directed* if it has no backward edge, and *simple* if all vertices v_i are different. When $k > 0$ and $v_0 = v_k$,

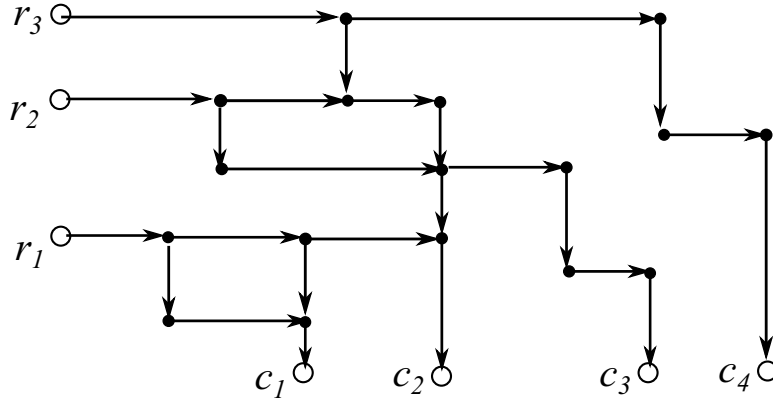
P is called a *cycle*, and called a *simple cycle* if, in addition, v_1, \dots, v_k are different. When it is not confusing, we may use for P the abbreviated notation via vertices: $P = v_0 v_1 \dots v_k$, or via edges: $P = e_1 e_2 \dots e_k$.

Also, using standard terminology in graph theory, for a directed edge $e = (u, v)$, we say that e *leaves* u and *enters* v , and that u is the *tail* and v is the *head* of e .

SE-graphs. A graph $G = (V, E)$ of this sort (also denoted as $(V, E; R, C)$) is defined by the following conditions:

- (i) G is planar (with a fixed layout in the plane);
- (ii) G has edges of two types: *horizontal* edges, or *H-edges*, which are directed from left to right, and *vertical* edges, or *V-edges*, which are directed downwards (so each edge points either *south* or *east*, justifying the term “SE-graph”);
- (iii) G has two distinguished subsets of vertices: set $R = \{r_1, \dots, r_m\}$ of *sources* and set $C = \{c_1, \dots, c_n\}$ of *sinks*; moreover, r_1, \dots, r_m are disposed on a vertical line, in this order upwards, and c_1, \dots, c_n are disposed on a horizontal line, in this order from left to right;
- (iv) each vertex (and each edge) of G belongs to a directed path from R to C .

The set $V - (R \cup C)$ of *inner* vertices of an SE-graph $G = (V, E)$ is denoted by $W = W_G$. An example of SE-graphs with $m = 3$ and $n = 4$ is drawn in the picture:



Each inner vertex $v \in W$ is regarded as an indeterminate (generator), and we assign a weight $w(e)$ to each edge e in a way similar to the assignment for Cauchon graphs in [2]. More precisely, for $e = (u, v) \in E$,

- (2.1) (i) $w(e) := v$ if e is an H-edge with $u \in R$;
- (ii) $w(e) := u^{-1}v$ if e is an H-edge with $u \in W$;
- (iii) $w(e) := 1$ if e is a V-edge.

This gives rise to defining the weight $w(P)$ of a directed path $P = e_1 e_2 \dots e_k$ in G , to be the ordered (from left to right) product

$$w(P) = w(e_1)w(e_2) \cdots w(e_k). \quad (2.2)$$

Then $w(P)$ is a Laurent monomial in elements of W . Note that when P begins in R and ends in C , its weight can also be expressed in the following useful form; cf. [3, Prop. 3.1.8]. Let $u_1, v_1, u_2, v_2, \dots, u_{d-1}, v_{d-1}, u_d$ be the sequence of vertices where P makes turns; namely, P changes the horizontal direction to the vertical one at each u_i , and conversely at each v_i . Then (due to the “telescopic effect” caused by (2.1)(ii)),

$$w(P) = u_1 v_1^{-1} u_2 v_2^{-1} \cdots u_{d-1} v_{d-1}^{-1} u_d. \quad (2.3)$$

We assume that the generators W obey (quasi)commutation laws somewhat similar to those for the quantum affine space (cf. (1.2)); namely,

(2.4) for distinct $u, v \in W$,

- (i) if there is a directed *horizontal* path from u to v in G , then $uv = qvu$;
- (ii) if there is a directed *vertical* path from u to v in G , then $vu = quv$;
- (iii) otherwise $uv = vu$.

Quantum minors. It will be convenient for us to visualize matrices in the Cartesian form: for an $m \times n$ matrix $A = (a_{ij})$, the row indices $i = 1, \dots, m$ are assumed to increase upwards, and the column indices $j = 1, \dots, n$ from left to right.

We denote by $A(I|J)$ the submatrix of A whose rows are indexed by $I \subseteq [m]$, and columns indexed by $J \subseteq [n]$. Let $|I| = |J| =: k$, and let I consist of $i_1 < \dots < i_k$ and J consist of $j_1 < \dots < j_k$. Then the q -determinant of $A(I|J)$, or the q -minor of A for $(I|J)$, is defined as

$$[I|J]_{A,q} := \sum_{\sigma \in S_k} (-q)^{\ell(\sigma)} \prod_{d=1}^k a_{i_d j_{\sigma(d)}}, \quad (2.5)$$

where, in the noncommutative case, the product under \prod is ordered by increasing d , and $\ell(\sigma)$ denotes the *length* (number of inversions) of a permutation σ . In the minor notation $[I|J]_{A,q}$, the terms A and/or q may be omitted when they are clear from the context.

Path matrices. An important construction in [2] associates to a Cauchon graph G a certain matrix, called the path matrix of G , which has a nice property of Lindström’s type: the q -minors of this matrix correspond to appropriate systems of disjoint paths in G .

This is extended to an arbitrary SE-graph $G = (V, E; R, C)$. More precisely, let $m := |R|$ and $n := |C|$. As before, $w = w_G$ denotes the edge weights in G defined by (2.1). For $i \in [m]$ and $j \in [n]$, we denote the set of directed paths from r_i to c_j in G by $\Phi_G(i|j)$.

Definition. The *path matrix* Path_G associated to G is the $m \times n$ matrix whose entries are defined by

$$\text{Path}_G(i|j) := \sum_{P \in \Phi_G(i|j)} w(P), \quad (i, j) \in [m] \times [n], \quad (2.6)$$

In particular, $\text{Path}_G(i|j) = 0$ if $\Phi_G(i|j) = \emptyset$.

Thus, the entries of Path_G belong to the \mathbb{K} -algebra \mathcal{L}_G of Laurent polynomials generated by the set W of inner vertices of G subject to relations (2.4). (Note also that Path_G is a q -matrix, i.e., its entries obey Manin's relations; see [4, Th. 3.2]).

Definition. Let $\mathcal{E}^{m,n}$ denote the set of pairs $(I|J)$ such that $I \subseteq [m]$, $J \subseteq [n]$ and $|I| = |J|$. Borrowing terminology from [5], we say that for $(I|J) \in \mathcal{E}^{m,n}$, a set ϕ of pairwise *disjoint* directed paths from the source set $R_I := \{r_i : i \in I\}$ to the sink set $C_J := \{c_j : j \in J\}$ in G is an $(I|J)$ -flow. The set of $(I|J)$ -flows is denoted by $\Phi(I|J) = \Phi_G(I|J)$.

We throughout assume that the paths forming ϕ are ordered by increasing the source indices. Namely, if I consists of $i(1) < i(2) < \dots < i(k)$ and J consists of $j(1) < j(2) < \dots < j(k)$, then ℓ -th path P_ℓ in ϕ begins at $r_{i(\ell)}$, and therefore, P_ℓ ends at $c_{j(\ell)}$ (which easily follows from the planarity of G , the ordering of sources and sinks in the boundary of G and the fact that the paths in ϕ are disjoint). We write $\phi = (P_1, P_2, \dots, P_k)$ and (similar to path systems in [2]) define the weight of ϕ to be the ordered product

$$w(\phi) := w(P_1)w(P_2) \cdots w(P_k). \quad (2.7)$$

Our first theorem is a direct extension of a q -analog of Lindström's Lemma shown for Cauchon graphs in [2, Th. 4.4]; it gives a relationship between flows and minors of path matrices.

Theorem 2.1 *Let G be an SE-graph with m sources and n sinks. Then for the path matrix $\text{Path} = \text{Path}_G$ and for any $(I|J) \in \mathcal{E}^{m,n}$, there holds*

$$[I|J]_{\text{Path},q} = \sum_{\phi \in \Phi(I|J)} w(\phi). \quad (2.8)$$

This theorem (stated in [4, Th. 3.1]) is proved in Section 5.

3 Double flows, matchings, and exchange operations

A study of quadratic identities for minors of quantum matrices in [4] is reduced to handling ordered products of minors of the path matrices of SE-graphs G , and further, in view of Theorem 2.1, to handling ordered pairs of flows in G . On this way, a crucial role is played by exchange operations on pairs of flows. To describe them, we first need some definitions and conventions.

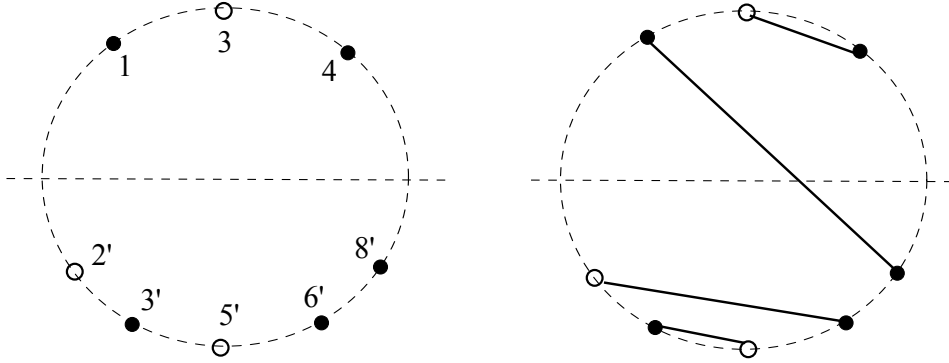
Let $G = (V, E; R, C)$ be an SE-graph with $|R| = m$ and $|C| = n$. For $(I|J), (I'|J') \in \mathcal{E}^{m,n}$, consider an $(I|J)$ -flow ϕ and an $(I'|J')$ -flow ϕ' in G . We call the ordered pair (ϕ, ϕ') a *double flow* in G . Define

$$\begin{aligned} I^\circ &:= I - I', & J^\circ &:= J - J', & I^\bullet &:= I' - I, & J^\bullet &:= J' - J, \\ Y^r &:= I^\circ \cup I^\bullet & \text{and} & & Y^c &:= J^\circ \cup J^\bullet. \end{aligned} \quad (3.1)$$

Note that $|I| = |J|$ and $|I'| = |J'|$ imply that $|Y^r| + |Y^c|$ is even and that

$$|I^\circ| - |I^\bullet| = |J^\circ| - |J^\bullet|. \quad (3.2)$$

It is convenient for us to interpret I° and I^\bullet as the sets of *white* and *black* elements of Y^r , respectively, and similarly for J°, J^\bullet, Y^c , and to visualize these objects by use of a *circular diagram* D in which the elements of Y^r (resp. Y^c) are disposed in the increasing order from left to right in the upper (resp. lower) half of a circumference O . For example if, say, $I^\circ = \{3\}$, $I^\bullet = \{1, 4\}$, $J^\circ = \{2', 5'\}$ and $J^\bullet = \{3', 6', 8'\}$, then the diagram is viewed as in the left fragment of the picture below. (Sometimes, to avoid a possible mess between elements of Y^r and Y^c , and when it leads to no confusion, we denote elements of Y^c with primes.)



We refer to the quadruple $(I|J, I'|J')$ as a *cortege*, and to $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ as the *refinement* of $(I|J, I'|J')$, or as a *refined cortege*.

Let M be a partition of $Y^r \sqcup Y^c$ into 2-element sets (recall that $A \sqcup B$ denotes the disjoint union of sets A, B). We refer to M as a *perfect matching* on $Y^r \sqcup Y^c$, and to its elements as *couples*.

Also we say that $\pi \in M$ is: an *R-couple* if $\pi \subseteq Y^r$, a *C-couple* if $\pi \subseteq Y^c$, and an *RC-couple* if $|\pi \cap Y^r| = |\pi \cap Y^c| = 1$ (as though π “links” two sources, two sinks, and one source and one sink, respectively).

Definition. A (perfect) matching M as above is called a *feasible* matching for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ if:

- (3.3) (i) for each $\pi = \{i, j\} \in M$, the elements i, j have different colors if π is an *R-couple* or a *C-couple*, and have the same color if π is an *RC-couple*;
- (ii) M is *planar*, in the sense that the chords connecting the couples in the circumference O are pairwise non-intersecting.

The right fragment of the above picture illustrates an instance of feasible matchings.

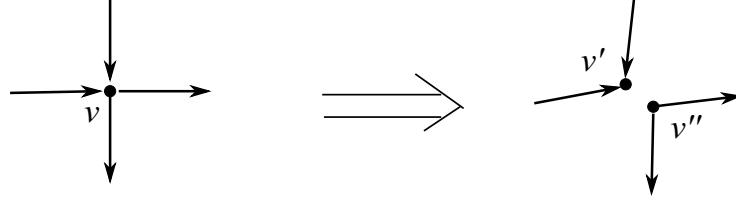
Return to a double flow (ϕ, ϕ') as above. We associate to it a feasible matching for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$ as follows. Let V_ϕ and E_ϕ , respectively, denote the sets of vertices and edges of G occurring in ϕ , and similarly for ϕ' . Denote by $\langle U \rangle$ the subgraph of G induced by the set of edges

$$U := E_\phi \Delta E_{\phi'},$$

writing $A \Delta B$ for the symmetric difference $(A - B) \cup (B - A)$ of sets A, B . Then

(3.4) a vertex v of $\langle U \rangle$ has degree 1 if $v \in R_{I^\circ} \cup R_{I^\bullet} \cup C_{J^\circ} \cup C_{J^\bullet}$, and degree 2 or 4 otherwise.

We slightly modify $\langle U \rangle$ by splitting each vertex v of degree 4 in $\langle U \rangle$ (if any) into two vertices v', v'' disposed in a small neighborhood of v so that the edges entering (resp. leaving) v become entering v' (resp. leaving v''); see the picture.



The resulting graph, denoted as $\langle U \rangle'$, is planar and has vertices of degree only 1 and 2. Therefore, $\langle U \rangle'$ consists of pairwise disjoint (non-directed) simple paths P'_1, \dots, P'_k (considered up to reversing) and, possibly, simple cycles Q'_1, \dots, Q'_d . The corresponding images of P'_1, \dots, P'_k (resp. Q'_1, \dots, Q'_d) give paths P_1, \dots, P_k (resp. cycles Q_1, \dots, Q_d) in $\langle U \rangle$. When $\langle U \rangle$ has vertices of degree 4, some of the latter paths and cycles may be self-intersecting and may “touch”, but not “cross”, each other. The following simple facts are shown in [4].

Lemma 3.1 (i) $k = (|I^\circ| + |I^\bullet| + |J^\circ| + |J^\bullet|)/2$;

(ii) the set of endvertices of P_1, \dots, P_k is $R_{I^\circ \cup I^\bullet} \cup C_{J^\circ \cup J^\bullet}$; moreover, each P_i connects either R_{I° and R_{I^\bullet} , or C_{J° and C_{J^\bullet} , or R_{I° and C_{J° , or R_{I^\bullet} and C_{J^\bullet} ;

(iii) in each path P_i , the edges of ϕ and the edges of ϕ' have different directions (say, the former edges are all forward, and the latter ones are all backward).

Thus, each P_i is represented as a concatenation $P_i^{(1)} \circ P_i^{(2)} \circ \dots \circ P_i^{(\ell)}$ of forwardly and backwardly directed paths which are alternately contained in ϕ and ϕ' . We call P_i an *exchange path* (by a reason that will be clear later). The endvertices of P_i determine, in a natural way, a pair of elements of $Y^r \sqcup Y^c$, denoted by π_i . Then $M := \{\pi_1, \dots, \pi_k\}$ is a perfect matching on $Y^r \sqcup Y^c$.

Moreover, M is a feasible matching for $(I^\circ, I^\bullet, J^\circ, J^\bullet)$, since property (3.3)(i) follows from Lemma 3.1(ii), and property (3.3)(ii) is provided by the fact that P'_1, \dots, P'_k are pairwise disjoint simple paths in $\langle U \rangle'$. We denote M as $M(\phi, \phi')$, and for $\pi \in M$, denote by $P(\pi)$ the exchange path P_i corresponding to π (i.e., $\pi = \pi_i$).

Figure 1 illustrates an instance of (ϕ, ϕ') for $I = \{1, 2, 3\}$, $J = \{1', 3', 4'\}$, $I' = \{2, 4\}$, $J' = \{2', 3'\}$; here ϕ and ϕ' are drawn by solid and dotted lines, respectively (in the left fragment), the subgraph $\langle E_\phi \Delta E_{\phi'} \rangle$ consists of three paths and one cycle (in the middle), and the circular diagram illustrates $M(\phi, \phi')$ (in the right fragment).

Ordinary flow exchange operation. Let us be given a double flow (ϕ, ϕ') for a cortege $(I|J, I'|J')$. Fix a couple $\pi = \{i, j\} \in M(\phi, \phi')$. The operation w.r.t. π rearranges (ϕ, ϕ') into another double flow (ψ, ψ') for some $(\tilde{I}|\tilde{J}, \tilde{I}'|\tilde{J}')$, as follows.

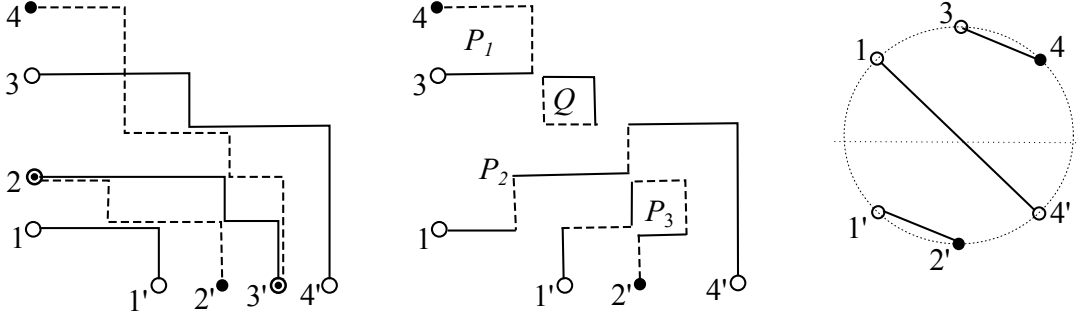


Figure 1: flows ϕ and ϕ' (left); $\langle E_\phi \Delta E_{\phi'} \rangle$ (middle); $M(\phi, \phi')$ (right)

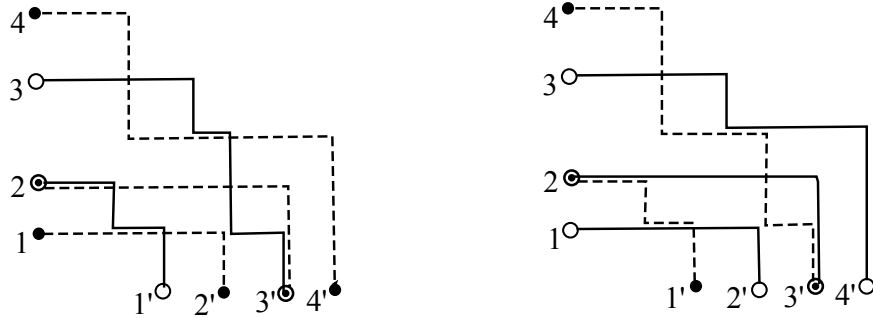
Consider the exchange path $P = P(\pi)$ corresponding to π , and let \mathcal{E} be the set of edges of P . Define

$$\tilde{I} := I \Delta (\pi \cap Y^r), \quad \tilde{I}' := I' \Delta (\pi \cap Y^r), \quad \tilde{J} := J \Delta (\pi \cap Y^c), \quad \tilde{J}' := J' \Delta (\pi \cap Y^c).$$

The following simple lemma is shown in [4].

Lemma 3.2 *The subgraph ψ induced by $E_\phi \Delta \mathcal{E}$ gives a $(\tilde{I}|\tilde{J})$ -flow, and the subgraph ψ' induced by $E_{\phi'} \Delta \mathcal{E}$ gives a $(\tilde{I}'|\tilde{J}')$ -flow in G . Furthermore, $E_\psi \cup E_{\psi'} = E_\phi \cup E_{\phi'}$, $E_\psi \Delta E_{\psi'} = E_\phi \Delta E_{\phi'} (= U)$, and $M(\psi, \psi') = M(\phi, \phi')$.*

We call the transformation $(\phi, \phi') \mapsto (\psi, \psi')$ in this lemma the *ordinary flow exchange operation* for (ϕ, ϕ') using $\pi \in M(\phi, \phi')$ (or using $P(\pi)$). Clearly a similar operation applied to (ψ, ψ') using the same π returns (ϕ, ϕ') . The picture below illustrates flows ψ, ψ' obtained from ϕ, ϕ' in Fig. 1 by the ordinary exchange operations using the path P_2 (left) and the path P_3 (right).



Now we formulate the second theorem of this paper; it will be proved in Section 6.

Theorem 3.3 *Let ϕ be an $(I|J)$ -flow, and ϕ' an $(I'|J')$ -flow in G . Let (ψ, ψ') be the double flow obtained from (ϕ, ϕ') by the ordinary flow exchange operation using a couple $\pi = \{f, g\} \in M(\phi, \phi')$. Then:*

(i) *when π is an R- or C-couple and $f < g$, we have*

$$\begin{aligned} w(\phi)w(\phi') &= qw(\psi)w(\psi') & \text{in case } f \in I \cup J; \\ w(\phi)w(\phi') &= q^{-1}w(\psi)w(\psi') & \text{in case } f \in I' \cup J'; \end{aligned}$$

(ii) *when π is an RC-couple, we have $w(\phi)w(\phi') = w(\psi)w(\psi')$.*

4 Commutation properties of paths

This section contains auxiliary lemmas that will be used in the proofs of Theorems 2.1 and 3.3. They deal with special pairs P, Q of paths in an SE-graph $G = (V, E; R, C)$ and compare the weights $w(P)w(Q)$ and $w(Q)w(P)$. Similar or close statements for Cauchon graphs are given in [2, 3], and our method of proof is somewhat similar and rather straightforward as well.

We need some terminology, notation and conventions.

When it is not confusing, vertices, edges, paths and other objects in G are identified with their corresponding images in the plane. We assume that the sets R and C lie on the coordinate rays $(0, \mathbb{R}_{\geq 0})$ and $(\mathbb{R}_{\geq 0}, 0)$, respectively (then G is disposed within $\mathbb{R}_{\geq 0}^2$). The coordinates of a point v in \mathbb{R}^2 (e.g., a vertex v of G) are denoted as $(\alpha(v), \beta(v))$. W.l.o.g. we may assume that two vertices $u, v \in V$ have the same first (second) coordinate if and only if they belong to a vertical (resp. horizontal) path in G , in which case u, v are called *V-dependent* (resp. *H-dependent*). When u, v are V-dependent, i.e., $\alpha(u) = \alpha(v)$, we say that u is *lower* than v (and v is *higher* than u) if $\beta(u) < \beta(v)$. (In this case the commutation relation $uv = qvu$ takes place.)

Let P be a path in G . We denote: the first and last vertices of P by s_P and t_P , respectively; the *interior* of P (the set of points of $P - \{s_P, t_P\}$ in \mathbb{R}^2) by $\text{Int}(P)$; the set of horizontal edges of P by E_P^H ; and the projection $\{\alpha(x) : x \in P\}$ by $\alpha(P)$. Clearly if P is directed, then $\alpha(P)$ is the interval between $\alpha(s_P)$ and $\alpha(t_P)$.

For a directed path P , the following are equivalent: P is non-vertical; $E_P^H \neq \emptyset$; $\alpha(s_P) \neq \alpha(t_P)$; we will refer to such a P as a *standard* path.

For a standard path P , we will take advantage from a compact expression for the weight $w(P)$. We call a vertex v of P *essential* if either P makes a turn at v (changing the direction from horizontal to vertical or back), or $v = s_P \notin R$ and the first edge of P is horizontal, or $v = t_P$ and the last edge of P is horizontal. If u_0, u_1, \dots, u_k is the sequence of essential vertices of P in the natural order, then the weight of P can be expressed as

$$w(P) = u_0^{\sigma_0} u_1^{\sigma_1} \dots u_k^{\sigma_k}, \quad (4.1)$$

where $\sigma_i = 1$ if P makes a \lceil -turn at u_i or if $i = k$, while $\sigma_i = -1$ if P makes a \lfloor -turn at u_i or if $i = 0$. (Compare with (2.3) where a path from R to C is considered.) It is easy to see that if P does not begin in R , then its essential vertices are partitioned into H-dependent pairs.

Throughout the rest of the paper, for brevity, we denote q^{-1} by \bar{q} , and for an inner vertex $v \in W$ regarded as a generator, we may denote v^{-1} by \bar{v} .

Now we start stating the desired lemmas on two directed paths P, Q . They deal with the case when P and Q are *weakly intersecting*, which means that

$$P \cap Q = \{s_P, t_P\} \cap \{s_Q, t_Q\}; \quad (4.2)$$

in particular, $\text{Int}(P) \cap \text{Int}(Q) = \emptyset$. For such P, Q , we say that P is *lower* than Q if there are points $x \in P$ and $y \in Q$ such that $\alpha(x) = \alpha(y)$ and $\beta(x) < \beta(y)$ (this

property does not depend on the choice of x, y). We define the value $\varphi = \varphi(P, Q)$ by the relation

$$w(P)w(Q) = \varphi w(Q)w(P).$$

Obviously, $\varphi(P, Q) = 1$ when P or Q is a V-path. In the lemmas below we default assume that both P, Q are standard.

Lemma 4.1 *Let $\{\alpha(s_P), \alpha(t_P)\} \cap \{\alpha(s_Q), \alpha(t_Q)\} \cap \mathbb{R}_{>0} = \emptyset$. Then $\varphi(P, Q) = 1$.*

Proof Consider an essential vertex u of P and an essential vertex v of Q . Then for any $\sigma, \sigma' \in \{1, -1\}$, we have $u^\sigma v^{\sigma'} = v^{\sigma'} u^\sigma$ unless u, v are dependent.

Suppose that u, v are V-dependent. From the hypotheses of the lemma it follows that at least one of the following is true: $\alpha(s_P) < \alpha(u) < \alpha(t_P)$, or $\alpha(s_Q) < \alpha(v) < \alpha(t_Q)$. For definiteness assume the former. Then there is another essential vertex z of P such that $\alpha(z) = \alpha(u) = \alpha(v)$. Moreover, P makes a \lrcorner -turn at one of u, z , and a \llcorner -turn at the other. Since $P \cap Q = \emptyset$ (in view of (4.2)), the vertices u, z are either both higher or both lower than v . Let for definiteness u, z occur in this order in P ; then $w(P)$ contains the terms u, \bar{z} . Let $w(Q)$ contain the term v^σ and let $uv^\sigma = \rho v^\sigma u$, where $\sigma \in \{1, -1\}$ and $\rho \in \{q, \bar{q}\}$. Then $\bar{z}v^\sigma = \bar{\rho}v^\sigma \bar{z}$, implying $u\bar{z}v^\sigma = v^\sigma u\bar{z}$. Hence the contributions to $w(P)w(Q)$ and $w(Q)w(P)$ from the pairs using terms u, z, v (namely $\{u, v^\sigma\}$ and $\{\bar{z}, v^\sigma\}$) are equal.

Next suppose that u, v are H-dependent. One may assume that $\alpha(u) < \alpha(v)$. Then Q contains one more essential vertex $y \neq v$ with $\beta(y) = \beta(v) = \beta(u)$. Also $\alpha(u) < \alpha(v)$ and $P \cap Q = \emptyset$ imply $\alpha(u) < \alpha(y)$. Let for definiteness $\alpha(y) < \alpha(v)$. Then $w(P)$ contains the terms \bar{y}, v , and we can conclude that the contributions to $w(P)w(Q)$ and $w(Q)w(P)$ from the pairs using terms u, y, v are equal (using the fact that $\alpha(u) < \alpha(y), \alpha(v)$).

These reasonings imply $\varphi(P, Q) = 1$. ■

Lemma 4.2 *Let $\alpha(s_P) = \alpha(s_Q) > 0$ and $\alpha(t_P) \neq \alpha(t_Q)$. Let P be lower than Q . Then $\varphi(P, Q) = q$.*

Proof Let u and v be the first essential vertices in P and Q , respectively. Then $\alpha(u) = \alpha(s_P) = \alpha(s_Q) = \alpha(v)$ (in view of $\alpha(s_P) = \alpha(s_Q) > 0$). Since P is lower than Q , we have $\beta(u) \leq \beta(v)$. Moreover, this inequality is strong (since $\beta(u) = \beta(v)$ is impossible in view of (4.2) and the obvious fact that u, v are the tails of first H-edges in P, Q , respectively).

Now arguing as in the above proof, we can conclude that the discrepancy between $w(P)w(Q)$ and $w(Q)w(P)$ can arise only due to swapping the vertices u, v . Since u gives the term \bar{u} in $w(P)$, and v the term \bar{v} in $w(Q)$, the contribution from these vertices to $w(P)w(Q)$ and $w(Q)w(P)$ are expressed as $\bar{u}\bar{v}$ and $\bar{v}\bar{u}$, respectively. Since $\beta(u) < \beta(v)$, we have $\bar{u}\bar{v} = q\bar{v}\bar{u}$, and the result follows. ■

Lemma 4.3 *Let $\alpha(t_P) = \alpha(t_Q)$ and let either $\alpha(s_P) \neq \alpha(s_Q)$ or $\alpha(s_P) = \alpha(s_Q) = 0$. Let P be lower than Q . Then $\varphi(P, Q) = q$.*

Proof We argue in spirit of the proof of Lemma 4.2. Let u and v be the last essential vertices in P and Q , respectively. Then $\alpha(u) = \alpha(t_P) = \alpha(t_Q) = \alpha(v)$. Also $\beta(u) < \beta(v)$ (since P is lower than Q , and in view of (4.2) and the fact that u, v are the heads of H-edges in P, Q , respectively). The condition on $\alpha(s_P)$ and $\alpha(s_Q)$ imply that the discrepancy between $w(P)w(Q)$ and $w(Q)w(P)$ can arise only due to swapping the vertices u, v (using reasonings as in the proof of Lemma 4.1). Observe that $w(P)$ contains the term u , and $w(Q)$ the term v . So the generators u, v contribute uv to $w(P)w(Q)$, and vu to $w(Q)w(P)$. Now $\beta(u) < \beta(v)$ implies $uv = qvu$, and the result follows. \blacksquare

Lemma 4.4 *Let $\alpha(t_P) = \alpha(s_Q)$ and $\beta(t_P) \geq \beta(s_Q)$. Then $\varphi(P, Q) = q$.*

Proof Let u be the last essential vertex in P and let v, z be the first and second essential vertices of Q , respectively (note that z exists because of $0 < \alpha(s_Q) < \alpha(t_Q)$). Then $\alpha(u) = \alpha(t_P) = \alpha(s_Q) = \alpha(v) < \alpha(z)$. Also $\beta(u) \geq \beta(t_P) \geq \beta(s_Q) \geq \beta(v) = \beta(z)$. Let Q' and Q'' be the parts of Q from s_Q to z and from z to t_Q , respectively. Then $\alpha(P) \cap \alpha(Q'') = \emptyset$, implying $\varphi_{P, Q''} = 1$ (using Lemma 4.1 when Q'' is standard). Hence $\varphi_{P, Q} = \varphi_{P, Q'}$.

To compute $\varphi_{P, Q'}$, consider three possible cases.

(a) Let $\beta(u) > \beta(v)$. Then u, v form the unique pair of dependent essential vertices for P, Q' . Note that $w(P)$ contains the term u , and $w(Q')$ contains the term \bar{v} . Since $\beta(u) > \beta(v)$, we have $u\bar{v} = q\bar{v}u$, implying $\varphi_{P, Q'} = q$.

(b) Let $u = v$ and let u be the unique essential vertex of P (in other words, P is an H-path with $s_P \in R$). Note that $u = v$ and $\beta(t_P) \geq \beta(s_Q)$ imply $t_Q = u = v = s_P$. Also $\alpha(u) < \alpha(z)$ and $\beta(u) = \beta(z)$; so u, z are dependent essential vertices for P, Q' and $uz = qzu$. We have $w(P) = u$ and $w(Q') = \bar{u}z$ (in view of $u = v$). Then $u\bar{u}z = \bar{u}uz = q\bar{u}zu$ gives $\varphi_{P, Q'} = q$.

(c) Now let $u = v$ and let y be the essential vertex of P preceding u . Then $t_Q = u = v = s_P$, $\beta(y) = \beta(u) = \beta(z)$, and $\alpha(y) < \alpha(u) < \alpha(z)$. Hence y, u, z are dependent, $w(P)$ contains $\bar{y}u$, and $w(Q') = \bar{u}z$. We have

$$\bar{y}u\bar{u}z = \bar{y}\bar{u}uz = (q\bar{u}\bar{y})(qzu) = q^2\bar{u}(\bar{q}z\bar{y})u = q\bar{u}z\bar{y}u,$$

again obtaining $\varphi_{P, Q'} = q$. \blacksquare

Lemma 4.5 *Let $\alpha(t_P) = \alpha(s_Q)$ and $\beta(t_P) < \beta(s_Q)$. Then $\varphi(P, Q) = \bar{q}$.*

Proof Let u be the last essential vertex of P , and v the first essential vertex of Q . Then $\alpha(u) = \alpha(t_P) = \alpha(s_Q) = \alpha(v)$, and $\beta(t_P) < \beta(s_Q)$ together with (4.2) implies $\beta(u) < \beta(v)$. Also $w(P)$ contains u and $w(Q)$ contains \bar{v} . Now $u\bar{v} = \bar{q}\bar{v}u$ implies $\varphi_{P, Q} = \bar{q}$. \blacksquare

5 Proof of Theorem 2.1

It can be conducted as a direct extension of the proof of a similar Lindström's type result given by Casteels [2, Sec. 4] for Cauchon graphs. To make our description more self-contained, we outline the main ingredients of the proof, leaving the details where needed to the reader.

Let $(I|J) \in \mathcal{E}^{m,n}$, $I = \{i(1) < \dots < i(k)\}$ and $J = \{j(1) < \dots < j(k)\}$. Recall that an $(I|J)$ -flow in an SE-graph G (with m sources and n sinks) consists of pairwise disjoint paths P_1, \dots, P_k from the source set $R_I = \{r_{i(1)}, \dots, r_{i(k)}\}$ to the sink set $C_J = \{c_{j(1)}, \dots, c_{j(k)}\}$, and (due to the planarity of G) we may assume that each P_d begins at $r_{i(d)}$ and ends at $c_{j(d)}$. Besides, we are forced to deal with an arbitrary *path system* $\mathcal{P} = (P_1, \dots, P_k)$ in which for $i = 1, \dots, k$, P_d is a directed path in G beginning at $r_{i(d)}$ and ending at $c_{j(\sigma(d))}$, where $\sigma(1), \dots, \sigma(k)$ are different, i.e., $\sigma = \sigma_{\mathcal{P}}$ is a permutation on $[k]$. (In particular, $\sigma_{\mathcal{P}}$ is identical if \mathcal{P} is a flow.)

We naturally partition the set of all path systems for G and $(I|J)$ into the set $\Phi = \Phi_G(I|J)$ of $(I|J)$ -flows and the rest $\Psi = \Psi_G(I|J)$ (consisting of those path systems that contain intersecting paths). The following property easily follows from the planarity of G (cf. [2, Lemma 4.2]):

(5.1) For any $\mathcal{P} = (P_1, \dots, P_k) \in \Psi$, there exist two *consecutive* intersecting paths P_d, P_{d+1} .

The q -sign of a permutation σ is defined by

$$\text{sgn}_q(\sigma) := (-q)^{\ell(\sigma)},$$

where $\ell(\sigma)$ is the length of σ (see Sect. 2).

Now we start computing the q -minor $[I|J]$ of the matrix Path_G with the following chain of equalities:

$$\begin{aligned} [I|J] &= \sum_{\sigma \in S_k} \text{sgn}_q(\sigma) \left(\prod_{d=1}^k \text{Path}_G(i(d)|j(\sigma(d))) \right) \\ &= \sum_{\sigma \in S_k} \text{sgn}_q(\sigma) \left(\prod_{d=1}^k \left(\sum (w(P) : P \in \mathcal{P}_G(i(d)|j(\sigma(d)))) \right) \right) \\ &= \sum (\text{sgn}_q(\sigma_{\mathcal{P}}) w(\mathcal{P}) : \mathcal{P} \in \Phi \cup \Psi) \\ &= \sum (w(\mathcal{P}) : \mathcal{P} \in \Phi) + \sum (\text{sgn}_q(\sigma_{\mathcal{P}}) w(\mathcal{P}) : \mathcal{P} \in \Psi). \end{aligned}$$

Thus, we have to show that the second sum in the last row is zero. It will follow from the existence of an involution $\eta : \Psi \rightarrow \Psi$ without fixed points such that for each $\mathcal{P} \in \Psi$,

$$\text{sgn}_q(\sigma_{\mathcal{P}}) w(\mathcal{P}) = -\text{sgn}_q(\sigma_{\eta(\mathcal{P})}) w(\eta(\mathcal{P})). \quad (5.2)$$

To construct the desired η , consider $\mathcal{P} = (P_1, \dots, P_k) \in \Psi$, take the minimal i such that P_i and P_{i+1} meet, take the last common vertex v of these paths, represent P_i as the concatenation $K \circ L$, and P_{i+1} as $K' \circ L'$, so that $t_K = t_{K'} = s_L = s_{L'} = v$, and

exchange the portions L, L' of these paths, forming $Q_i := K \circ L'$ and $Q_{i+1} := K' \circ L$. Then we assign $\eta(\mathcal{P})$ to be obtained from \mathcal{P} by replacing P_i, P_{i+1} by Q_i, Q_{i+1} . It is routine to check that η is indeed an involution (with $\eta(\mathcal{P}) \neq \mathcal{P}$) and that

$$\ell(\sigma_{\eta(\mathcal{P})}) = \ell(\sigma_{\mathcal{P}}) + 1, \quad (5.3)$$

assuming w.l.o.g. that $\sigma(i) < \sigma(i+1)$. On the other hand, applying to the paths K, L, K', L' corresponding lemmas from Sect. 4 (among Lemmas 4.2–4.4), one can obtain

$$\begin{aligned} w(P_i)w(P_{i+1}) &= w(K)w(L)w(K')w(L') = qw(K)w(L)w(L')w(K') \\ &= q^2w(K)w(L')w(L)w(K') = qw(K)w(L')w(K')w(L) = qw(Q_i)w(Q_{i+1}), \end{aligned}$$

whence $w(\mathcal{P}) = qw(\eta(\mathcal{P}))$. This together with (5.3) gives

$$\text{sgn}_q(\sigma_{\mathcal{P}})w(\mathcal{P}) + \text{sgn}_q(\sigma_{\eta(\mathcal{P})})w(\eta(\mathcal{P})) = (-q)^{\ell(\sigma_{\mathcal{P}})}qw(\eta(\mathcal{P})) + (-q)^{\ell(\sigma_{\mathcal{P}})+1}w(\eta(\mathcal{P})) = 0,$$

yielding (5.2), and the result follows. ■

6 Proof of Theorem 3.3

Using notation as in the hypotheses of this theorem, we first consider the case when

(C): $\pi = \{f, g\}$ is a C -couple in $M(\phi, \phi')$ with $f < g$ and $f \in J$.

(Then $f \in J^\circ$ and $g \in J^\bullet$.) We have to prove that

$$w(\phi)w(\phi') = qw(\psi)w(\psi') \quad (6.1)$$

The proof is given throughout Sects. 6.1–6.5. The other possible cases in Theorem 3.3 will be discussed in Sect. 6.6.

6.1 Snakes and links. Let Z be the exchange path determined by π (i.e., $Z = P(\pi)$ in notation of Sect. 3). It connects the sinks c_f and c_g , which may be regarded as the first and last vertices of Z , respectively. Then Z is representable as a concatenation $Z = \overline{Z}_1 \circ Z_2 \circ \overline{Z}_3 \circ \dots \circ \overline{Z}_{k-1} \circ Z_k$, where k is even, each Z_i with i odd (even) is a directed path concerning ϕ (resp. ϕ'), and \overline{Z}_i stands for the path reversed to Z_i . More precisely, let $z_0 := c_f$, $z_k := c_g$, and for $i = 1, \dots, k-1$, denote by z_i the common endvertex of Z_i and Z_{i+1} . Then each Z_i with i odd is a directed path from z_i to z_{i-1} in $\langle E_\phi - E_{\phi'} \rangle$, while each Z_i with i even is a directed path from z_{i-1} to z_i in $\langle E_{\phi'} - E_\phi \rangle$.

We refer to Z_i with i odd (even) as a *white* (resp. *black*) *snake*.

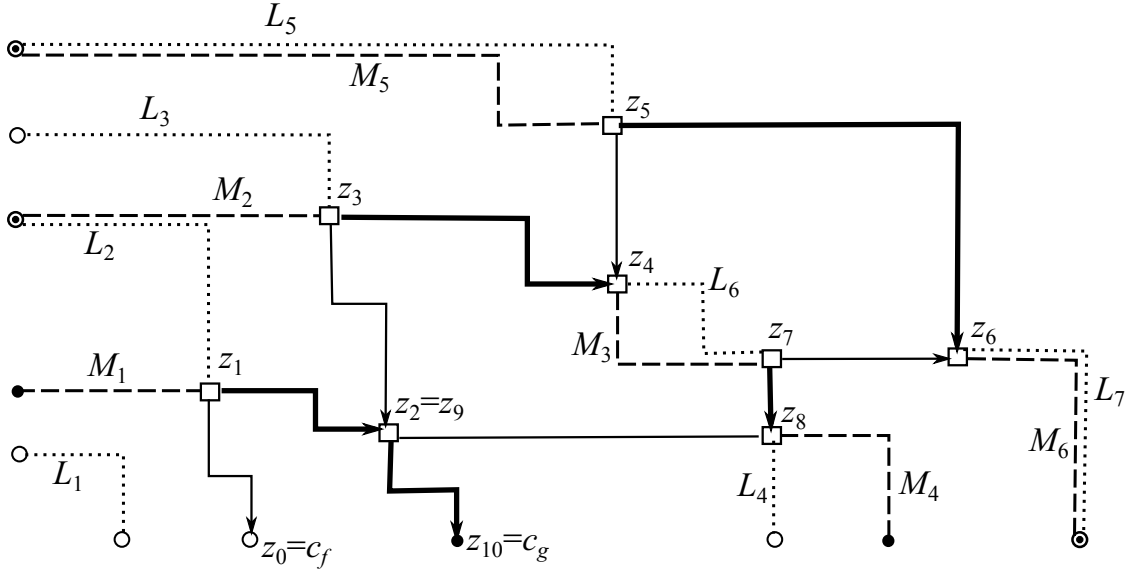
Also we refer to the vertices z_1, \dots, z_{k-1} as the *bends* of Z . A bend z_i is called a *peak* (a *pit*) if both path Z_i, Z_{i+1} leave (resp. enter) z_i ; then z_1, z_3, \dots, z_{k-1} are the peaks, and z_2, z_4, \dots, z_{k-2} are the pits. Note that some peak z_i and pit z_j may coincide; in this case we say that z_i, z_j are *twins*.

The rests of flows ϕ and ϕ' consist of directed paths that we call *white* and *black links*, respectively. More precisely, the white (black) links correspond to the connected components of the subgraph ϕ (resp. ϕ') from which the interiors of all snakes are removed. So a link connects either (a) a source and a sink (being a component of ϕ or ϕ'), or (b) a source and a pit, or (c) a peak and a sink, or (d) a pit and a peak. We say that a link is *unbounded* in case (a), *semi-bounded* in cases (b),(c), and *bounded* in case (d). Note that

- (6.2) a bend z_i occurs as an endvertex in exactly four paths among snakes and links, namely: either in two snakes and two links (of different colors), or in four snakes $Z_i, Z_{i+1}, Z_j, Z_{j+1}$ (when z_i, z_j are twins).

We denote the sets of snakes and links (for ϕ, ϕ', π) by \mathcal{S} and \mathcal{L} , respectively; the corresponding subsets of white and black elements of these sets are denoted as $\mathcal{S}^\circ, \mathcal{S}^\bullet, \mathcal{L}^\circ, \mathcal{L}^\bullet$.

The picture below illustrates an example. Here $k = 10$, the bends z_1, \dots, z_9 are marked by squares, the white and black snakes are drawn by thin and thick solid zigzag lines, respectively, the white links (L_1, \dots, L_7) by short-dotted lines, and the black links (M_1, \dots, M_6) by long-dotted lines.



The weight $w(\phi)w(\phi')$ of the double flow (ϕ, ϕ') can be written as the corresponding ordered product of the weights of snakes and links; let \mathcal{N} be the string (sequence) of snakes and links in this product. The weight of the double flow (ψ, ψ') uses a string consisting of the same snakes and links but taken in another order; we denote this string by \mathcal{N}^* .

We say that two elements among snakes and links are *invariant* if they occur in the same order in \mathcal{N} and \mathcal{N}^* , and *permuting* otherwise. In particular, two links of different colors are invariant, whereas two snakes of different colors are always permuting.

For example, observe that the string \mathcal{N} for the above illustration is viewed as

$$L_1 L_2 Z_1 L_3 Z_3 Z_9 L_4 L_5 Z_5 L_6 Z_7 L_7 M_1 Z_2 Z_{10} M_2 Z_4 M_3 Z_8 M_4 M_5 Z_6 M_6,$$

whereas \mathcal{N}^* is viewed as

$$L_1 L_2 Z_2 Z_{10} L_3 Z_4 L_6 Z_8 L_4 L_5 Z_6 L_7 M_1 Z_1 M_2 Z_3 Z_9 M_4 M_5 Z_5 M_3 Z_7 M_6.$$

For $A, B \in \mathcal{S} \cup \mathcal{L}$, we write $A \prec B$ (resp. $A \prec^* B$) if A occurs in \mathcal{N} (resp. in \mathcal{N}^*) earlier than B . We define $\varphi_{A,B} = \varphi_{B,A} := 1$ if A, B are invariant, and define $\varphi_{A,B} = \varphi_{B,A}$ by the relation

$$w(A)w(B) = \varphi_{A,B}w(B)w(A). \quad (6.3)$$

if A, B are permuting and $A \prec B$. Note that $\varphi_{A,B}$ is defined somewhat differently than $\varphi(P, Q)$ in Sect. 4.

For $A, B \in \mathcal{S} \cup \mathcal{L}$, we may use notation (A, B) when A, B are permuting and $A \prec B$ (and may write $\{A, B\}$ when their orders by \prec and \prec^* are not important for us).

Our goal is to prove that in case (C),

$$\prod (\varphi_{A,B} : A, B \in \mathcal{S} \cup \mathcal{L}) = q, \quad (6.4)$$

whence (6.1) will immediately follow.

We first consider the *non-degenerate* case. This means the following restriction:

(6.5) all coordinates $\alpha(z_1), \dots, \alpha(z_{k-1}), \alpha(c_1), \dots, \alpha(c_n)$ of bends and sinks are different.

The proof of (6.4) subject to (6.5) will consist of three stages I, II, III where we compute the total contribution from the pairs of links, the pairs of snakes, and the pairs consisting of one snake and one link, respectively. As a consequence, the following three results will be obtained (implying (6.4)).

Proposition 6.1 *In case (6.5), the product φ^I of the values $\varphi_{A,B}$ over links $A, B \in \mathcal{L}$ is equal to 1.*

Proposition 6.2 *In case (6.5), the product φ^{II} of the values $\varphi_{A,B}$ over snakes $A, B \in \mathcal{S}$ is equal to q .*

Proposition 6.3 *In case (6.5), the product φ^{III} of the values $\varphi_{A,B}$ where one of A, B is a snake and the other is a link is equal to 1.*

These propositions are proved in Sects. 6.2–6.4. Sometimes it will be convenient for us to refer to a white (black) snake/link concerning ϕ, ϕ', π as a ϕ -snake/link (resp. a ϕ' -snake/link), and similarly for ψ, ψ', π .

6.2 Proof of Proposition 6.1. Under the exchange operation using Z , any ϕ -link becomes a ψ -link and any ϕ' -link becomes a ψ' -link. The white links occur in \mathcal{N} earlier than the black links, and similarly for \mathcal{N}^* . Therefore, if A, B are permuting links, then they are of the same color. This implies that $A \cap B = \emptyset$. Also each endvertex of any link either is a bend or belongs to $R \cup C$. Then (6.5) implies that the sets $\{\alpha(s_A), \alpha(t_A)\} \cap \mathbb{R}_{>0}$ and $\{\alpha(s_B), \alpha(t_B)\} \cap \mathbb{R}_{>0}$ are disjoint. Now Lemma 4.1 gives $\varphi_{A,B} = 1$, and the proposition follows. \blacksquare

6.3 Proof of Proposition 6.2. Consider two snakes $A = Z_i$ and $B = Z_j$, and let $A \prec B$. If $|i-j| > 1$ then $A \cap B = \emptyset$ and, moreover, $\{\alpha(s_A), \alpha(t_A)\} \cap \{\alpha(s_B), \alpha(t_B)\} = \emptyset$ (since Z is simple and in view of (6.5)). This gives $\varphi_{A,B} = 1$, by Lemma 4.1.

Now let $|i-j| = 1$. Then A, B have different colors; hence A is white and B is black (in view of $A \prec B$). So i is odd, and two cases are possible:

Case 1: $j = i + 1$ and z_i is a peak: $z_i = s_A = s_B$;

Case 2: $j = i - 1$ and z_{i-1} is a pit: $z_{i-1} = t_A = t_B$.

Cases 1,2 are divided into two subcases each.

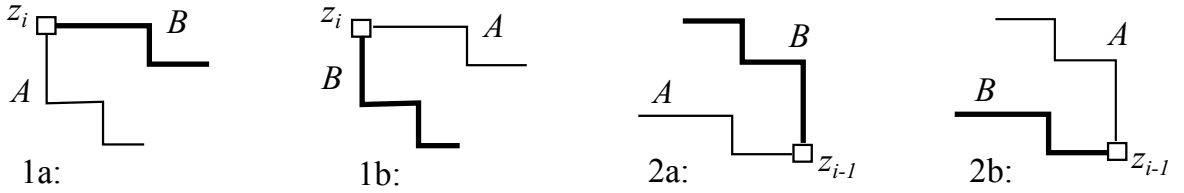
Subcase 1a: $j = i + 1$ and A is lower than B .

Subcase 1b: $j = i + 1$ and B is lower than A .

Subcase 2a: $j = i - 1$ and A is lower than B .

Subcase 2b: $j = i - 1$ and B is lower than A .

(Recall that for directed paths P, Q satisfying (4.2), P is said to be *lower* than Q if there are $x \in P$ and $y \in Q$ with $\alpha(x) = \alpha(y)$ and $\beta(x) < \beta(y)$.) Subcases 1a–2b are illustrated in the picture:



Under the exchange operation using Z , any snake changes its color; so A, B are permuting. Applying to A, B Lemmas 4.2 and 4.3, we obtain $\varphi_{A,B} = q$ in Subcases 1a,2a, and $\varphi_{A,B} = \bar{q}$ in Subcases 1b,2b.

It is convenient to associate with a bend z the number $\gamma(z)$ which is equal to $+1$ if, for the corresponding pair $A \in \mathcal{S}^\circ$ and $B \in \mathcal{S}^\bullet$ sharing z , A is lower than B (as in Subcases 1a,2a), and equal to -1 otherwise (as in Subcases 1b,2b). Define

$$\gamma_Z := \sum (\gamma(z) : z \text{ a bend of } Z). \quad (6.6)$$

Then $\varphi^{II} = q^{\gamma_Z}$. Thus, $\varphi^{II} = q$ is equivalent to

$$\gamma_Z = 1. \quad (6.7)$$

To show (6.7), we are forced to deal with a more general setting. More precisely, let us turn Z into simple cycle D by combining the directed path Z_1 (from z_1 to $z_0 = c_f$) with the horizontal path from c_f to c_g (to create the latter, we formally add to G the horizontal edges (c_j, c_{j+1}) for $j = f, \dots, g-1$). The resulting directed path \tilde{Z} from z_1 to $c_g = z_k$ is regarded as the new white snake replacing Z_1 . Then \tilde{Z}_1 shares the end z_k with the black path Z_k ; so z_k is a pit of D , and \tilde{Z} is lower than Z_k . Thus, compared with Z , the cycle D acquires an additional bend, namely, z_k . We have $\gamma(z_k) = 1$, implying $\gamma_D = \gamma_Z + 1$. Then (6.7) is equivalent to $\gamma_D = 2$.

On this way, we come to a new (more general) setting by considering an arbitrary simple (non-directed) cycle D rather than a special path Z . Moreover, instead of an SE-graph as before, we can work with a more general directed planar graph G in which any edge $e = (u, v)$ points arbitrarily within the south-east sector, i.e., satisfies $\alpha(u) \leq \alpha(v)$ and $\beta(u) \geq \beta(v)$. We call G of this sort a *weak SE-graph*.

So now we are given a colored simple cycle D in G , i.e., D is representable as a concatenation $\overline{D}_1 \circ D_2 \circ \dots \circ \overline{D}_{k-1} \circ D_k$, where each D_i is a directed path in G ; a path (snake) D_i with i odd (even) is colored white (resp. black). Let d_1, \dots, d_k be the sequence of bends in D , i.e., d_i is a common endvertex of D_{i-1} and D_i (letting $D_0 := D_k$). We assume that D is oriented according to the direction of D_i with i even. When this orientation is clockwise (counterclockwise) around a point in the open bounded region O_D of the plane surrounded by D , we say that D is *clockwise* (resp. *counterclockwise*). In particular, the cycle arising from the above path Z is clockwise.

Our goal is to prove the following

Lemma 6.4 *Let D be a colored simple cycle in a weak SE-graph G . If D is clockwise then $\gamma_D = 2$. If D is counterclockwise then $\gamma_D = -2$.*

Proof We use induction on the number $\eta(D)$ of bends of D . It suffices to consider the case when D is clockwise (since for a counterclockwise cycle $D' = \overline{D}'_1 \circ D'_2 \circ \dots \circ \overline{D}'_{k-1} \circ D'_k$, the reversed cycle $\overline{D}' = \overline{D}'_k \circ D'_{k-1} \circ \dots \circ \overline{D}'_2 \circ D'_1$ is clockwise, and it is easy to see that $\gamma_{\overline{D}'} = -\gamma_{D'}$).

W.l.o.g., one may assume that the coordinates $\beta(d_i)$ of all bends d_i are different (as we can make, if needed, a due small perturbation on D , which does not affect γ).

If $\eta(D) = 2$, then $D = \overline{D}_1 \circ D_2$, and the clockwise orientation of D implies that the path D_1 is lower than D_2 . So $\gamma(d_1) = \gamma(d_2) = 1$, implying $\gamma_D = 2$.

Now assume that $\eta(D) > 2$. Then at least one of the following is true:

- (a) there exists a peak d_i such that the horizontal line through d_i meets D on the left of d_i , i.e., there is a point x in D with $\alpha(x) < \alpha(d_i)$ and $\beta(x) = \beta(d_i)$;
- (b) there exists a pit d_i such that the horizontal line through d_i meets D on the right of d_i .

(This can be seen as follows. Let d_j be a peak with $\beta(d_j)$ maximum. If $\beta(d_{j-1}) \leq \beta(d_{j+1})$, then, by easy topological reasonings, either the pit d_{j+1} is as required in (b) (when d_{j+2} is on the right from D_{j+1}), or the peak d_{j+2} is as required in (a) (when d_{j+2}

is on the left from D_{j+1}), or both. And if $\beta(d_{j-1}) > \beta(d_{j+1})$, similar properties hold for d_{j-1} and d_{j+2} .)

We may assume that case (a) takes place (for case (b) is symmetric to (a)). Choose the point x as in (a) with $\alpha(x)$ maximum and draw the horizontal line-segment L connecting the points x and d_i . Then the interior of L does not meet D . Two cases are possible:

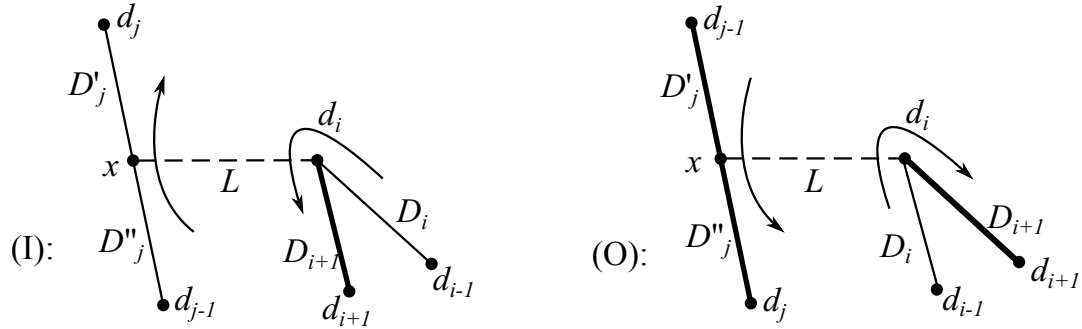
(I) $\text{Int}(L)$ is contained in the region O_D ; or

(O) $\text{Int}(L)$ is outside O_D .

Since x cannot be a bend of D (in view of $\beta(x) = \beta(d_i)$ and $\beta(d_i) \neq \beta(d_{i'})$ for any $i' \neq i$), x is an interior point of some snake D_j ; let D'_j and D''_j be the parts of D_j from s_{D_j} to x and from x to t_{D_j} , respectively. Using the facts that D is oriented clockwise and this orientation is agreeable with the forward (backward) direction of each black (resp. white) snake, one can conclude that

(6.8) (a) in case (I), D_j is white and $\gamma(d_i) = -1$ (i.e., for the white snake D_i and black snake D_{i+1} that share the peak d_i , D_{i+1} is lower than D_i); and (b) in case (O), D_j is black and $\gamma(d_i) = 1$ (i.e., D_i is lower than D_{i+1})

See the picture (where the orientation of D is indicated):

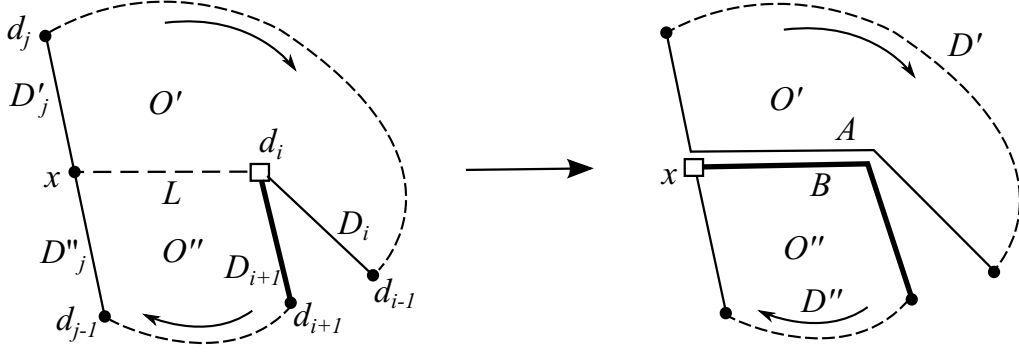


The points x and d_i split the cycle (closed curve) D into two parts ζ', ζ'' , where the former contains D'_j and the latter does D''_j .

We first examine case (I). The line L divides the region O_D into two parts O' and O'' lying above and below L , respectively. Orienting the curve ζ' from x to d_i and adding to it the segment L oriented from d_i to x , we obtain closed curve D' surrounding O' . Note that D' is oriented clockwise around O' . We combine the paths D'_j , L (from x to d_i) and D_i into one directed path A (going from $s_{D'_j} = s_{D_j} = d_j$ to $t_{D_i} = d_{i-1}$). Then D' turns into a correctly colored simple cycle in which A is regarded as a white snake and the white/black snakes structure on the rest preserves (cf. (6.8)(a)).

In its turn, the curve ζ'' oriented from d_i to x plus the segment L (oriented from x to d_i) form closed curve D'' that surrounds O'' and is oriented clockwise as well. We combine L and D_{i+1} into one black snake B (going from x to d_{i+1}). Then D'' becomes a correctly colored cycle, and x is a peak in it. (The point x turns into a vertex of G .) We have $\gamma(x) = 1$ (since the white D''_j is lower than the black B).

The creation of D', D'' from D in case (I) is illustrated in the picture:



We observe that, compared with D , the pair D', D'' misses the bend d_i (with $\gamma(d_i) = -1$) but acquires the bend x (with $\gamma(d) = 1$). Then

$$\eta(D) = \eta(D') + \eta(D''), \quad (6.9)$$

implying $\eta(D'), \eta(D'') < \eta(D)$. Therefore, we can apply induction. This gives $\gamma_{D'} = \gamma_{D''} = 2$. Now, by reasonings above,

$$\gamma_D = \gamma_{D'} + \gamma_{D''} + \gamma(d_i) - \gamma(x) = 2 + 2 - 1 - 1 = 2,$$

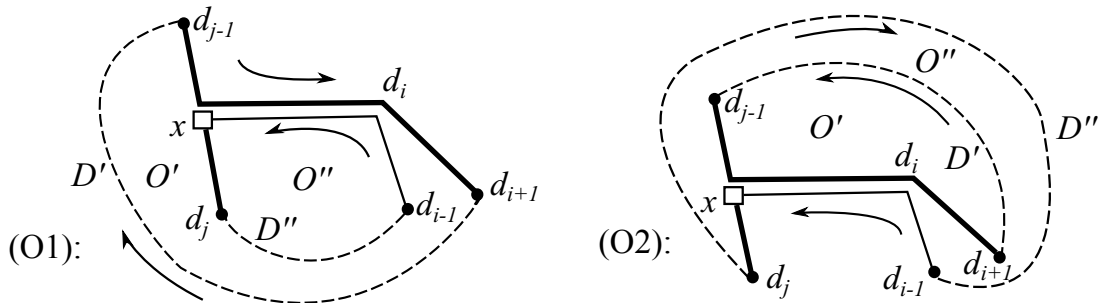
as required.

Next we examine case (O). From the fact that D simple one can conclude that the curve ζ' (containing D'_j) passes through the black snake D_{i+1} , and the curve ζ'' (containing D''_j) through the white snake D_i . Adding to each of ζ', ζ'' a copy of L , we obtain closed curves D', D'' , respectively, each inheriting the orientation of D . They become correctly colored simple cycles when we combine the paths D'_j, L, D_{i+1} into one black snake (from d_{j-1} to d_{i+1}) in D' , and combine the paths L, D_i into one white snake (from the new bend x to d_i) in D'' . Let O', O'' be the bounded regions in the plane surrounded by D', D'' , respectively. It is not difficult topological exercise to see that two cases are possible:

(O1) O' includes O'' (and O_D);

(O2) O'' includes O' (and O_D).

These cases are illustrated in the picture:



Then in case (O1), D' is clockwise and D'' is counterclockwise, whereas in case (O2) the behavior is converse. Also $\gamma(d_i) = 1$ and $\gamma(x) = -1$. Similar to case (I), (6.9) is

true and we can apply induction. Then in case (O1), we have $\gamma_{D'} = 2$ and $\gamma_{D''} = -2$, whence

$$\gamma_D = \gamma_{D'} + \gamma_{D''} + \gamma(d_i) - \gamma(x) = 2 - 2 + 1 - (-1) = 2.$$

And in case (O2), we have $\gamma_{D'} = -2$ and $\gamma_{D''} = 2$, whence

$$\gamma_D = \gamma_{D'} + \gamma_{D''} + \gamma(d_i) - \gamma(x) = -2 + 2 + 1 - (-1) = 2.$$

Thus, in all cases we obtain $\gamma_D = 2$, yielding the lemma. ■

This completes the proof of Proposition 6.2. ■

6.4 Proof of Proposition 6.3. Consider a link L . By Lemma 4.1, for any snake P , $\varphi_{L,P} \neq 1$ is possible only if L and P have a common endvertex v . Note that $v \notin R \cup C$. In particular, it suffices to examine only bounded and semi-bounded links.

First assume that $s_L \notin R$. Then there are exactly two snakes containing s_L , namely, a white snake A and a black snake B such that $s_L = t_A = t_B$. If L is white, then A and L belong to the same path in ϕ ; therefore, $A \prec L \prec B$. Under the exchange operation A becomes black, B becomes white, and L continues to be white. Then B, L belong to the same path in ψ ; this implies $B \prec^* L \prec^* A$. So both pairs (A, L) and (L, B) are permuting, and Lemma 4.4 gives $\varphi_{A,L} = q$ and $\varphi_{L,B} = \bar{q}$, whence $\varphi_{A,L}\varphi_{L,B} = 1$.

Now let L be black. Then $A \prec B \prec L$ and $B \prec^* A \prec^* L$. So both pairs $\{A, L\}$ and $\{B, L\}$ are invariant, whence $\varphi_{A,L} = \varphi_{B,L} = 1$.

The end t_L is examined in a similar way. Assuming $t_L \notin C$, there are exactly two snakes, a white snake A' and a black snake B' , that contain t_L , namely: $t_L = s_{A'} = s_{B'}$. If L is white, then $L \prec A' \prec B'$ and $L \prec^* B' \prec^* A'$. Therefore, $\{L, A'\}$ and $\{L, B'\}$ are invariant, yielding $\varphi_{L,A'} = \varphi_{L,B'} = 1$. And if L is black, then $A' \prec L \prec B'$ and $B' \prec^* L \prec^* A'$. So both (A', L) and (L, B') are permuting, and we obtain from Lemma 4.4 that $\varphi_{A',L} = \bar{q}$ and $\varphi_{L,B'} = q$, yielding $\varphi_{A',L}\varphi_{L,B'} = 1$.

These reasonings prove the proposition. ■

6.5 Degenerate case. We have proved relation (6.4) in a non-degenerate case, i.e., subject to (6.5), and now our goal is to prove (6.4) when the set

$$\mathcal{Z} := \{z_1, \dots, z_{k-1}\} \cup \{c_j : j \in J \cup J'\}$$

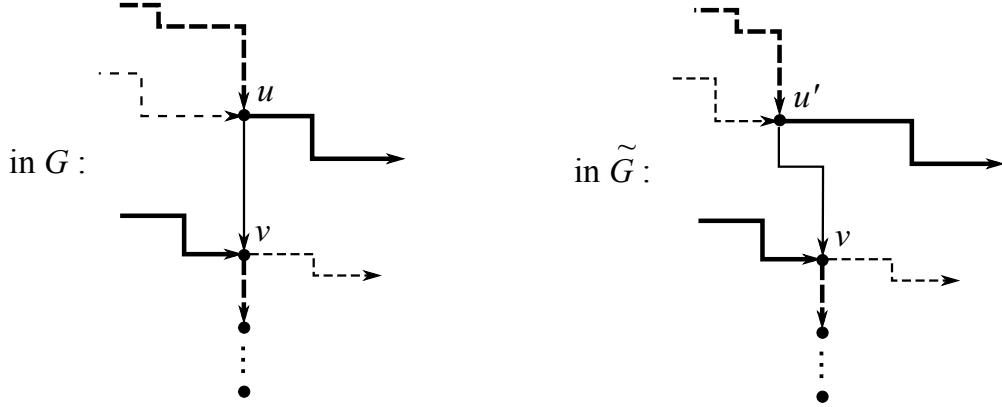
contains distinct elements u, v with $\alpha(u) = \alpha(v)$. We say that such u, v form a *defect pair*. A special defect pair is formed by twins z_i, z_j (bends satisfying $i \neq j$, $\alpha(z_i) = \alpha(z_j)$ and $\beta(z_i) = \beta(z_j)$). Another special defect pair is of the form $\{s_P, t_P\}$ when P is a *vertical* snake or link, i.e., $\alpha(s_P) = \alpha(t_P)$.

We will show (6.4) by induction on the number of defect pairs.

Let a be the *minimum* number such that the set $X := \{u \in \mathcal{Z} : \alpha(u) = a\}$ contains a defect pair. We denote the elements of X as v_0, v_1, \dots, v_r , where for each i , v_{i-1} is *higher* than v_i , which means that either $\beta(v_{i-1}) > \beta(v_i)$, or v_{i-1}, v_i are twins and v_{i-1}

is a pit (while v_i is a peak) in the exchange path Z . The highest element v_0 in this order is also denoted by u .

In order to conduct induction, we deform the graph G within a sufficiently narrow vertical strip $S = [a - \epsilon, a + \epsilon] \times \mathbb{R}$ (where $0 < \epsilon < \min\{|\alpha(z) - a| : z \in \mathcal{Z} - X\}$) to get rid of the defect pairs involving u in such a way that the configuration of snakes/links in the arising graph \tilde{G} remains “equivalent” to the initial one. More precisely, we shift the bend u at a small distance ($< \epsilon$) to the left, keeping the remaining elements of \mathcal{Z} ; then the bend u' arising in place of u satisfies $\alpha(u') < \alpha(u)$ and $\beta(u') = \beta(u)$. The snakes/links with an endvertex at u are transformed accordingly; see the picture for an example.



Let Π and $\tilde{\Pi}$ denote the L.H.S. value in (6.4) for the initial and new configurations, respectively. Under the deformation, the number of defect pairs becomes smaller, so we may assume by induction that $\Pi = q$. Thus, we have to prove that

$$\Pi = \tilde{\Pi}. \quad (6.10)$$

We need some notation and conventions. For $v \in X$, the set of (initial) snakes and links with an endvertex at v is denoted by \mathcal{P}_v . For $U \subseteq X$, \mathcal{P}_U denotes $\cup(\mathcal{P}_v : v \in U)$. Corresponding objects for the deformed graph \tilde{G} are usually denoted with tildes as well; e.g.: for a path P in G , its image in \tilde{G} is denoted by \tilde{P} ; the image of \mathcal{P}_v is denoted by $\tilde{\mathcal{P}}_v$ (or $\tilde{\mathcal{P}}_{\tilde{v}}$), and so on. The set of standard paths in \mathcal{P}_U (resp. $\tilde{\mathcal{P}}_U$) is denoted by $\mathcal{P}_U^{\text{st}}$ (resp. $\tilde{\mathcal{P}}_U^{\text{st}}$). Define

$$\Pi_{u, X-u} := \prod (\varphi_{P, Q} : P \in \mathcal{P}_u, Q \in \mathcal{P}_{X-u}). \quad (6.11)$$

A similar product for \tilde{G} (i.e., with $\tilde{\mathcal{P}}_u$ instead of \mathcal{P}_u) is denoted by $\tilde{\Pi}_{u, X-u}$.

Note that (6.10) is equivalent to

$$\Pi_{u, X-u} = \tilde{\Pi}_{u, X-u}. \quad (6.12)$$

This follows from the fact that for any paths $P, Q \in \mathcal{S} \cup \mathcal{L}$ different from those involved in (6.11), the values $\varphi_{P, Q}$ and $\varphi_{\tilde{P}, \tilde{Q}}$ are equal. (The only nontrivial case arises when $P, Q \in \mathcal{P}_u$ and Q is vertical (so \tilde{Q} becomes standard). Then $t_Q = v_1$. Hence $Q \in \mathcal{P}_{X-u}$, the pair P, Q is involved in $\Pi_{u, X-u}$, and the pair \tilde{P}, \tilde{Q} in $\tilde{\Pi}_{u, X-u}$.)

To simplify our description technically, one trick will be of use. Suppose that for each standard path $P \in \mathcal{P}_X^{\text{st}}$, we choose a point (not necessarily a vertex) $v_P \in \text{Int}(P)$ in such a way that $\alpha(s_P) < \alpha(v_P) < \alpha(t_P)$, and the coordinates $\alpha(v_P)$ for all such paths P are different. Then v_P splits P into two subpaths P', P'' , where we denote by P' the subpath connecting s_P and v_P when $\alpha(s_P) = a$, and connecting v_P and t_P when $\alpha(t_P) = a$, while P'' is the rest. This provides the following property: for any $P, Q \in \mathcal{P}_X^{\text{st}}$, $\varphi_{P', Q''} = \varphi_{Q', P''} = 1$ (in view of Lemma 4.1). Hence $\varphi_{P, Q} = \varphi_{P', Q'} \varphi_{P'', Q''}$. Also $P'' = \tilde{P}''$. It follows that (6.12) would be equivalent to the equality

$$\prod (\varphi_{P', Q'} : P \in \mathcal{P}_u, Q \in \mathcal{P}_{X-\{u\}}) = \prod (\varphi_{\tilde{P}', \tilde{Q}'} : P \in \mathcal{P}_u, Q \in \mathcal{P}_{X-\{u\}}).$$

In light of these reasonings, it suffices to prove (6.12) in the special case when

$$(6.13) \text{ any } P \in \mathcal{P}_u \text{ and } Q \in \mathcal{P}_{X-u} \text{ satisfy } \{\alpha(s_P), \alpha(t_P)\} \cap \{\alpha(s_Q), \alpha(t_Q)\} = \{a\}.$$

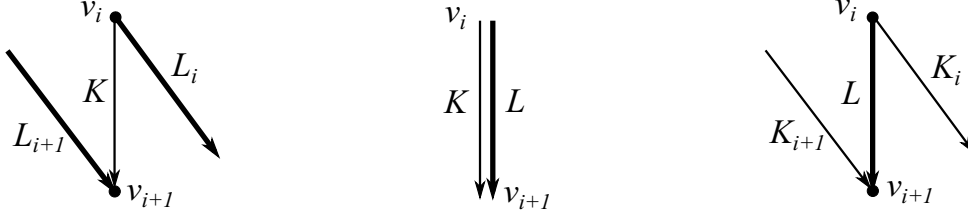
For $i = 0, \dots, r$, we denote by A_i, B_i, K_i, L_i , respectively, the white snake, black snake, white link, and black link, that have an endvertex at v_i . Note that if v_{i-1}, v_i are twins, then the fact that v_{i-1} is a pit implies A_{i-1}, B_{i-1} are the snakes entering v_{i-1} , and A_i, B_i are the snakes leaving v_i ; for convenience, we formally define $K_{i-1} = K_i$ and $L_{i-1} = L_i$ to be the trivial paths consisting of the the same single vertex v_i . Note that if $v_r \in C$, then some paths among A_k, B_k, K_k, L_k vanish (e.g., both snakes and one link).

When vertices v_i and v_{i+1} are connected by a (vertical) path in $\mathcal{S} \cup \mathcal{L}$, we denote such a path by P_i and say that the vertex v_i is *open*; otherwise v_i is said to be closed. Note that v_i, v_{i+1} can be connected by either one snake, or one link, or two links (namely, K_i, L_i); in the latter case P_i is chosen arbitrarily among them. In particular, if v_i, v_{i+1} are twins, then v_i is open and the role of P_i is played by any of the trivial links K_i, L_i . Obviously, in a sequence of vertical paths P_i, P_{i+1}, \dots, P_j , the snakes and links alternate. One can see that if P_i is a white snake, i.e., $P_i = A_i = A_{i+1} =: A$, then both black snakes B_i, B_{i+1} are standard, and there holds $v_i = s_{B_i}$ and $v_{i+1} = t_{B_{i+1}}$. See the left fragment of the picture:



Symmetrically, if P_i is a black snake: $B_i = B_{i+1} =: B$, then the white snakes A_i, A_{i+1} are standard, $v_i = s_{A_i}$ and $v_{i+1} = t_{A_{i+1}}$; see the right fragment of the above picture.

In its turn, if P_i is a nontrivial white link, i.e., $P_i = K_i = K_{i+1}$, then two cases are possible: either the black links L_i, L_{i+1} are standard, $v_i = s_{L_i}$ and $v_{i+1} = t_{L_{i+1}}$, or $L_i = L_{i+1} = P_i$. And if P_i is a black link, the behavior is symmetric. See the picture:



Now we are ready to start proving equality (6.12). Note that the deformation of G changes none of the orders \prec and \prec^* .

We say that paths $P, P' \in \mathcal{P}_X^{\text{st}}$ are *separated* (from each other) if they are not contained in the same path of any of the flows ϕ, ϕ', ψ, ψ' . The following observation will be of use:

(6.14) if $P, P' \in \mathcal{P}_X^{\text{st}}$ have the same color, are separated, and P' is lower than P , then $P' \prec P$; and similarly w.r.t. the order \prec^* (concerning ψ, ψ').

Indeed, suppose that P, P' are white, and let Q and Q' be the paths of the flow ϕ containing P and P' , respectively. Since P, P' are separated, the paths Q, Q' are different. Moreover, the fact that P' is lower than P implies that Q' is lower than Q (taking into account that Q, Q' are disjoint). Thus, Q' precedes Q in ϕ , yielding $P' \prec P$, as required. When P, P' concern one of ϕ', ψ, ψ' , the argument is similar.

In what follows we will use the abbreviated notation A, B, K, L for the paths A_0, B_0, K_0, L_0 (respectively) having an endvertex at $u = v_0$. Also for $R \in \mathcal{P}_{X-u}$, we denote the product $\varphi_{A,R} \varphi_{B,R} \varphi_{K,R} \varphi_{L,R}$ by $\Pi(R)$, and denote by $\tilde{\Pi}(R)$ a similar product for the paths $\tilde{A}, \tilde{B}, \tilde{K}, \tilde{L}, \tilde{R}$ (concerning the deformed graph \tilde{G}). One can see that $\Pi_{u, X-u}$ (resp. $\tilde{\Pi}_{u, X-u}$) is equal to the product of the values $\Pi(R)$ (resp. $\tilde{\Pi}(R)$) over $R \in \mathcal{P}_{X-u}$.

To show (6.12), we will examine several cases. First of all we consider

Case (R1): $\{u\}$ is closed; in other words, all paths A, B, K, L are standard (taking into account that u is the highest vertex in X).

Proposition 6.5 *In case (R1), $\Pi(R) = \tilde{\Pi}(R) = 1$ holds for any $R \in \mathcal{P}_{X-u}$. As a consequence, (6.12) is valid.*

Proof Let $R \in \mathcal{P}_{v_p}$ for $p \geq 1$. Observe that (6.13) together with the fact that the vertex u is shifted under the deformation of G implies that $\{\alpha(s_{\tilde{P}}), \alpha(t_{\tilde{P}})\} \cap \{\alpha(s_{\tilde{R}}), \alpha(t_{\tilde{R}})\} = \emptyset$ holds for any $P \in \mathcal{P}_u$. This gives $\tilde{\Pi}(R) = 1$, by Lemma 4.1.

Next we show the equality $\Pi(R) = 1$. One may assume that R is standard (otherwise the equality is trivial). It is easy to see that in case (R1), each of A, B, K, L is separated from R .

Note that A, B, K, L, R are as follows: either (a) $t_A = t_B = s_K = s_L$ or (b) $s_A = s_B = t_K = t_L$, and either (c) $\alpha(s_R) = a$ or (d) $\alpha(t_R) = a$. Let us examine the possible cases when the combination of (a) and (d) takes place.

1) Let R be a white link, i.e., $R = K_p$. Since R is white and lower than A, B, K, L , we have $R \prec A, B, K, L$ (cf. (6.14)). Under the exchange operation (which, as we know, changes the colors of snakes and preserves the colors of links), R remains white. Then $R \prec^* A, B, K, L$. Therefore, all pairs $\{P, R\}$ with $P \in \mathcal{P}_u$ are invariant, and $\Pi(R) = 1$ is trivial.

2) Let $R = L_p$. Since R is black, we have $A, K \prec R \prec B, L$. The exchange operation changes the colors of A, B and preserves the ones of K, L, R . Hence $B, K \prec^* R \prec^* A, L$, giving the permuting pairs (A, R) and (R, B) . Lemma 4.3 applied to these pairs implies $\varphi_{A,R} = \bar{q}$ and $\varphi_{R,B} = q$. Then $\Pi(R) = \varphi_{A,R}\varphi_{R,B} = \bar{q}q = 1$.

3) Let $R = A_p$. Then $R \prec A, B, K, L$ and $B, K \prec^* R \prec^* A, L$ (since the exchange operation changes the colors of A, B, R but not K, L). This gives the permuting pairs (R, B) and (R, K) . Then $\varphi_{R,B} = q$, by Lemma 4.3, and $\varphi_{R,K} = \bar{q}$ by Lemma 4.5, and we have $\Pi(R) = \varphi_{R,B}\varphi_{R,K} = 1$.

4) Let $R = B_p$. (In fact, this case is symmetric to the previous one, as it is obtained by swapping (ϕ, ϕ') and (ψ, ψ') . Yet we prefer to give a proof in detail.) We have $A, K \prec R \prec B, L$ and $R \prec^* A, B, K, L$, giving the permuting pairs (A, R) and (K, R) . Then $\varphi_{A,R} = \bar{q}$, by Lemma 4.3, and $\varphi_{K,R} = q$, by Lemma 4.5, whence $\Pi(R) = 1$.

The other combinations, namely, (a) and (c), (b) and (c), (b) and (d), are examined in a similar way (where we appeal to appropriate lemmas from Sect. 4, and we leave this to the reader as an exercise. \blacksquare)

Next we consider

Case (R2): u is open; in other words, at least one path among A, B, K, L is vertical (going from u to v_1).

It falls into several subcases examined in propositions below.

Proposition 6.6 *In case (R2), let $R \in \mathcal{P}_{X-u}^{\text{st}}$ be separated from A, B, K, L . Then $\Pi(R) = \tilde{\Pi}(R)$.*

Proof We first assume that $u = v_0$ and v_1 are connected by exactly one path P_0 (which may be any of A, B, K, L) and give a reduction to the previous proposition, as follows.

Suppose that we replace P_0 by a standard path P' of the same color and type (snake or link) such that $s_{P'} = u$ (and $\alpha(t_{P'}) < a$). Then the set $\mathcal{P}'_u := (\{A, B, K, L\} - \{P_0\}) \cup \{P'\}$ becomes as in case (R1), and by Proposition 6.5, the corresponding product $\Pi'(R)$ of values $\varphi_{R,P}$ over $P \in \mathcal{P}'_u$ is equal to 1. (This relies on the fact that R is separated from A, B, K, L , which implies validity of (6.12) for R and corresponding $P \in \mathcal{P}'_u$.)

Now compare the effects from P' and \tilde{P}_0 . These paths have the same color and type, and both are separated from, and higher than R . Also $\alpha(s_{P'}) = \alpha(t_{\tilde{P}_0}) = a$ (since $s_{P'} = u$ and $t_{\tilde{P}_0} = v_1$). Then using appropriate lemmas from Sect. 4, one can conclude that $\{\varphi_{R,P'}, \varphi_{R,\tilde{P}_0}\} = \{q, \bar{q}\}$. Therefore,

$$\tilde{\Pi}(R) = \varphi_{R,\tilde{P}_0} = \Pi'(R)\varphi_{R,P'}^{-1} = \Pi(R).$$

Now let u and v_1 be connected by two paths, namely, by K, L . We again can appeal to Proposition 6.5. Consider $\mathcal{P}_u'' := \{A, B, K'', L''\}$, where K'', L'' are standard links (white and black, respectively) with $s_{K''} = s_{L''} = u$. Then $\Pi''(R) := \Pi(\varphi_{R,P} : P \in \mathcal{P}_u'') = 1$ and $\{\varphi_{R,K''}, \varphi_{R,\tilde{K}}\} = \{\varphi_{R,L''}, \varphi_{R,\tilde{L}}\} = \{q, \bar{q}\}$, and we obtain

$$\tilde{\Pi}(R) = \varphi_{R,\tilde{K}}\varphi_{R,\tilde{L}} = \Pi''(R)\varphi_{R,K''}^{-1}\varphi_{R,L''}^{-1} = \varphi_{R,A}\varphi_{R,B} = \Pi(R),$$

as required. ■

Proposition 6.7 *In case (R2), let R be a standard path in \mathcal{P}_{v_p} with $p \geq 1$. Let R be not separated from at least one of A, B, K, L . Then $\Pi(R) = \tilde{\Pi}(R)$.*

Proof We first assume that P_0 is the unique vertical path connecting u and v_i (in particular, u and v_1 are not twins). Then R is not separated from P_0 .

Suppose that P_0 and R are contained in the same path of the flow ϕ ; equivalently, both P_0, R are white and $P_0 \prec R$. Then neither ψ nor ψ' has a path containing both P_0, R (this is easy to conclude from the fact that one of R and P_{p-1} is a snake and the other is a link). Consider four possible cases for P_0, R .

(a) Let both P_0, R be links, i.e., $P_0 = K$ and $R = K_p$. Then $A, K \prec K_p \prec B, L$ and $K_p \prec^* B, K, A, L$ (since $K \prec^* K_p$ is impossible by the above observation). This gives the permuting pairs (A, K_p) and (\tilde{K}, K_p) , yielding $\varphi_{A,K_p} = \varphi_{\tilde{K},K_p}$.

(b) Let $P_0 = K$ and $R = A_p$. Then $A, K \prec A_p \prec B, L$ and $B, K \prec^* A_p \prec^* A, L$. This gives the permuting pairs (A, A_p) and (A_p, B) , yielding $\varphi_{A,A_p}\varphi_{\tilde{A}_p,B} = 1 = \varphi_{\tilde{K},A_p}$.

(c) Let $P_0 = A$ and $R = K_p$. Then $K, A \prec K_p \prec L, B$ and $K_p \prec^* K, B, L, A$. This gives the permuting pairs (K, K_p) and (\tilde{A}, K_p) , yielding $\varphi_{K,K_p} = \varphi_{\tilde{A},K_p}$.

(d) Let $P_0 = A$ and $R = A_p$. Then $K, A \prec A_p \prec L, B$ and $K, B \prec^* A_p \prec^* L, A$. This gives the permuting pairs (\tilde{A}, A_p) and (\tilde{A}_p, B) , yielding $\varphi_{\tilde{A},A_p} = \varphi_{A_p,B}$.

In all cases, we obtain $\Pi(R) = \tilde{\Pi}(R)$.

When P_0, R are contained in the same path in ϕ' (i.e., P_0, R are black and $P_0 \prec R$), we argue in a similar way. The cases with P_0, R contained in the same path of ψ or ψ' are symmetric.

A similar analysis is applicable (yielding $\Pi(R) = \tilde{\Pi}(R)$) when u and v_1 are connected by two vertical paths (namely, K, L) and exactly one relation among $K \prec R$, $L \prec R$, $K \prec^* R$ and $L \prec^* R$ takes place (equivalently: either K, R or L, R are separated, not both).

Finally, let u and v_1 be connected by both K, L , and assume that K, R are not separated, and similarly for L, R . An important special case is when $p = 1$ and u, v_1 are twins.

Note that from the assumption it easily follows that R is a snake. If R is the white snake A_p , then we have $A, K \prec A_p \prec B, L$ and $B, K, A, L \prec^* A_p$. This gives the permuting pairs (A, A_p) and (\tilde{K}, A_p) , yielding $\varphi_{A,A_p} = \varphi_{\tilde{K},A_p}$ (since $\alpha(t_A) = \alpha(t_{\tilde{K}})$). The case with $R = B_p$ is symmetric. In both cases, $\Pi(R) = \tilde{\Pi}(R)$. ■

Proposition 6.8 *Let $R = P_0$ be the unique vertical path connecting u and v_1 . Then $\Pi(R) = \tilde{\Pi}(R) = 1$.*

Proof The equality $\Pi(R) = 1$ is trivial. To see $\tilde{\Pi}(R) = 1$, consider possible cases for R . If $R = K$, then $\tilde{A} \prec \tilde{K} \prec \tilde{B}, \tilde{L}$ and $\tilde{B} \prec^* \tilde{K} \prec^* \tilde{A}, \tilde{L}$, giving the permuting pairs (\tilde{A}, \tilde{K}) and (\tilde{K}, \tilde{B}) (note that $t_{\tilde{A}} = t_{\tilde{B}} = s_{\tilde{K}} = \tilde{u}$). If $R = L$, then $\tilde{A}, \tilde{K}, \tilde{B} \prec \tilde{L}$ and $\tilde{B}, \tilde{K}, \tilde{A} \prec^* \tilde{L}$; so all pairs involving \tilde{L} are invariant. If $R = A$, then $\tilde{K} \prec \tilde{A} \prec \tilde{L}, \tilde{B}$ and $\tilde{K}, \tilde{B}, \tilde{L} \prec^* \tilde{A}$, giving the permuting pairs (\tilde{A}, \tilde{L}) and (\tilde{A}, \tilde{B}) (note that $s_{\tilde{A}} = s_{\tilde{B}} = t_{\tilde{L}} = \tilde{u}$). And the case $R = B$ is symmetric to the previous one.

In all cases, using appropriate lemmas from Sect. 4 (and relying on the fact that all paths $\tilde{A}, \tilde{B}, \tilde{K}, \tilde{L}$ are standard), one can conclude that $\tilde{\Pi}(R) = 1$. ■

Proposition 6.9 *Let both K, L be vertical. Then $\Pi(K)\Pi(L) = \tilde{\Pi}(K)\tilde{\Pi}(L) = 1$.*

Proof The equality $\Pi(K)\Pi(L) = 1$ is trivial. To see $\tilde{\Pi}(K)\tilde{\Pi}(L) = 1$, observe that $\tilde{A} \prec \tilde{K} \prec \tilde{B} \prec \tilde{L}$ and $\tilde{B} \prec^* \tilde{K} \prec^* \tilde{A} \prec^* \tilde{L}$. This gives the permuting pairs (\tilde{A}, \tilde{K}) and (\tilde{K}, \tilde{B}) . Using Lemma 4.4, we obtain $\varphi_{\tilde{A}, \tilde{K}} = q$ and $\varphi_{\tilde{K}, \tilde{B}} = \bar{q}$, and the result follows. ■

Taken together, Propositions 6.6–6.9 embrace all possibilities in case (R2). Adding to them Proposition 6.5 concerning case (R1), we easily obtain the desired relation (6.12) in a degenerate case.

This completes the proof of Theorem 3.3 in case (C), namely, relation (6.1). ■■

6.6 Other cases. Let $(I|J), (I'|J'), \phi, \phi', \psi, \psi'$ and $\pi = \{f, g\}$ be as in the hypotheses of Theorem 3.3. We have proved this theorem in case (C), i.e., when π is a C -couple with $f < g$ and $f \in J$ (see the beginning of Sect. 6). In other words, the exchange path $Z = P(\pi)$, used to transform the initial double flow (ϕ, ϕ') into the new double flow (ψ, ψ') , connects the sinks c_f and c_g that are covered by the “white flow” ϕ and the “black flow” ϕ' , respectively.

The other possible cases in the theorem are as follows:

- (C1) π is a C -couple with $f < g$ and $f \in J'$;
- (C2) π is an R -couple with $f < g$ and $f \in I$;
- (C3) π is an R -couple with $f < g$ and $f \in I'$;
- (C4) π is an RC -couple with $f \in I$ and $g \in J$;
- (C5) π is an RC -couple with $f \in I'$ and $g \in J'$.

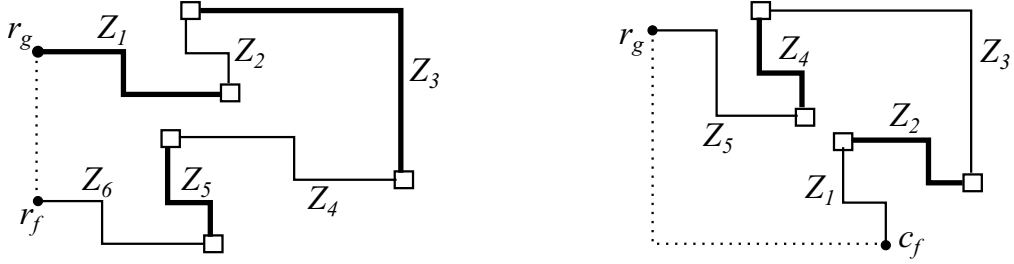
Case (C1) is symmetric to (C). This means that if double flows (ϕ, ϕ') and (ψ, ψ') are obtained from each other by applying the exchange operation using π (which, in particular, changes the “colors” of both f and g), and if one double flow is subject to (C) (i.e., f concerns the first, “white”, flow), then the other is subject to (C1) (i.e., f concerns the second, “black”, flow). Rewriting $w(\phi)w(\phi') = qw(\psi)w(\psi')$ (cf. (6.1)) as $w(\psi)w(\psi') = q^{-1}w(\phi)w(\phi')$, we just obtain the required equality in case (C1) (where (ψ, ψ') and (ϕ, ϕ') play the roles of the initial and updated double flows, respectively).

For a similar reasons, case (C3) is symmetric to (C2), and (C5) is symmetric to (C4). So it suffices to establish the desired equalities merely in cases (C2) and (C4).

To do this, we appeal to reasonings similar to those in Sects. 6.2–6.5. More precisely, it is not difficult to see that descriptions in Sects. 6.2 and 6.4 (concerning link-link and snake-link pairs in \mathcal{N}) remain applicable and Propositions 6.1 and 6.3 are directly extended to cases (C2) and (C4). The method of getting rid of degeneracies developed in Sect. 6.5 does work, without any troubles, for (C2) and (C4) as well.

As to the method in Sect 6.3 (concerning snake-snake pairs in case (C)), it should be modified as follows. We use terminology and notation from Sects. 6.1 and 6.3 and appeal to Lemma 6.4.

When dealing with case (C2), we represent the exchange path $Z = P(\pi)$ as a concatenation $Z_1 \circ \overline{Z}_2 \circ Z_3 \circ \cdots \circ \overline{Z}_k$, where each Z_i with i odd (even) is a snake contained in the black flow ϕ' (resp. the white flow ϕ). Then Z_1 begins at the source r_g and Z_k begins at the source r_f . An example with $k = 6$ is illustrated in the left fragment of the picture:



The common vertex (bend) of Z_i and Z_{i+1} is denoted by z_i . As before, we associate with a bend z the number $\gamma(z)$ (equal to 1 if, in the pair of snakes sharing z , the white snake is lower than the black one, and -1 otherwise), and define γ_Z as in (6.6). We turn Z into simple cycle D by combining the directed path Z_k (from r_f to z_{k-1}) with the vertical path from r_g to r_f , which is formally added to G . (In the above picture, this path is drawn by a dotted line.) Then, compared with Z , the cycle D has an additional bend, namely, r_g . Since the extended white path \tilde{Z}_k is lower than the black path Z_1 , we have $\gamma(r_g) = 1$, and therefore $\gamma_D = \gamma_Z + 1$.

One can see that the cycle D is oriented clockwise (where, as before, the orientation is defined according to that of black snakes). So $\gamma_D = 2$, by Lemma 6.4, implying $\gamma_Z = 1$. This is equivalent to the “snake-snake relation” $\varphi^{II} = q$, and as a consequence, we obtain the desired equality

$$w(\phi)w(\phi') = qw(\psi)w(\psi').$$

Finally, in case (C4), we represent the exchange path Z as the corresponding concatenation $\overline{Z}_1 \circ Z_2 \circ \overline{Z}_3 \circ \cdots \circ Z_{k-1} \circ \overline{Z}_k$ (with k odd), where the first white snake Z_1 ends at the sink c_f and the last white snake Z_k begins at the source r_g . See the right fragment of the above picture, where $k = 5$. We turn Z into simple cycle D by adding a new “black snake” Z_{k+1} beginning at r_g and ending at c_f (it is formed by the vertical

path from r_g to $(0, 0)$, followed by the horizontal path from $(0, 0)$ to c_f ; see the above picture). Compared with Z , the cycle D has two additional bends, namely, r_g and c_f . Since the black snake Z_{k+1} is lower than both Z_1 and Z_k , we have $\gamma(r_g) = \gamma(c_f) = -1$, whence $\gamma_D = \gamma_Z - 2$. Note that the cycle D is oriented counterclockwise. Therefore, $\gamma_D = -2$, by Lemma 6.4, implying $\gamma_Z = 0$. As a result, we obtain the desired equality $w(\phi)w(\phi') = w(\psi)w(\psi')$.

This completes the proof of Theorem 3.3.

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